The Poynting vector: power and energy in electromagnetic fields

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Conservation of energy in electromagnetics

The concept of conservation of energy (along with conservation of momentum) is one of the basic principles of physics – both classical and modern. When dealing with electromagnetic fields, a way is needed to relate the concept of energy to the fields. This is done by means of the Poynting vector: \( \mathbf{P} = \mathbf{E} \times \mathbf{H} \). In eq. (1) \( \mathbf{E} \) is the electric field intensity, \( \mathbf{H} \) is the magnetic field intensity, and \( \mathbf{P} \) is the Poynting vector, which is found to be the power density in the electromagnetic field. The conservation of energy is then established by means of the Poynting theorem.

◆◆◆ The Poynting theorem:

By using the Maxwell equations for the curl of the fields along with Gauss’s divergence theorem and an identity from vector analysis, we may prove what is known as the Poynting theorem. The Maxwell’s equations needed are

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (2)
\]

\[
\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad (3)
\]

along with the material relationships

\[
\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \quad (4)
\]

\[
\mathbf{B} = \mu_0 \mathbf{H} + \mu_0 \mathbf{M} \quad (5)
\]

or for isotropic materials

\[
\mathbf{D} = \varepsilon \mathbf{E} \quad (6)
\]

\[
\mathbf{B} = \mu \mathbf{H} \quad (7)
\]

In addition, the identity from vector analysis,
The derivation of $P$ is needed. The power density, then its surface integral over the surface of a volume must be the power out of the volume. So next do the negative of the surface integral to obtain $P$, the power into the volume: $P = -\mathbf{E} \times \mathbf{H} \cdot dS$. (9)

Next, substitute from Maxwell's equations, eqs. (2) and (3), in eq. (10) to obtain

$$P = -\iint \mathbf{E} \times \mathbf{H} \cdot dS = \iiint \mathbf{E} \cdot \mathbf{J} dv + \iiint \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} dv + \iiint \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} dv. \quad (11)$$

Eq. (11) is the Poynting theorem.

**Interpretation of the Poynting theorem:**

The interpretation of eq. (11) in terms of stored energies and power flows, to satisfy the conservation of energy, requires one to make some assumptions on the nature of the material where the fields are located. Let us assume that the fields are in an isotropic medium where the values of $\epsilon$ and $\mu$ are independent of time. Then we may use eqs. (6) and (7) in eq. (11) and move and $\mu$ into and out of the time derivative to obtain

$$P = \iiint \mathbf{E} \cdot \mathbf{J} dv + \frac{d}{dt} \iiint \left( \frac{1}{2} \varepsilon |\mathbf{E}|^2 + \frac{1}{2} \mu |\mathbf{H}|^2 \right) dv. \quad (12)$$

(In the above equation, use has been made of the fact that $d(\mathbf{E} \cdot \mathbf{E})/dt = d(E \cdot E)/dt$.)

**Energy density in electromagnetic fields:**
Conservation of energy is satisfied by eq. (12) provided one identifies $\frac{1}{2} |\vec{E}|^2$ as the energy density in the electric field and $\frac{1}{2} \mu |\vec{H}|^2$ as the energy density in the magnetic field. Then eq. (12), in words, is “Power into a volume from electromagnetic fields is equal to the power converted into heat in the volume plus the increase in the energy stored in the fields in the volume.”

A last comment - one make a play on words: the Poynting vector is pointing in the direction of power flow.

**Examples:**

Consider a coaxial cable having an inner conductor of radius $a$ and an outer conductor (of inside) radius $b$. If this cable carries a current $I$ and has a potential difference $V$ between the conductors, then from circuit theory the power down the cable is $VI$.

Now consider the fields in the dielectric of the cable. First the electric field may be obtained from Gauss’s law:

$$\vec{E} = \frac{\rho_v}{2\pi \epsilon \rho} \hat{\rho}$$

This field may be related to $V$ by $V = \int_a^b \vec{E} \cdot d\vec{l}$ to obtain

$$V = \frac{\rho_v}{2\pi \epsilon} \ln \left( \frac{b}{a} \right)$$

which in eq. (13) gives

$$\vec{E} = \frac{V}{\ln \left( \frac{b}{a} \right)} \hat{\rho}$$

Next, Ampere’s circuital law gives for the magnetic field intensity,

$$\vec{H} = \frac{I}{2\pi \rho} \hat{\phi}$$

Now form $P = \vec{E} \times \vec{H} = \frac{VI}{2\pi \rho^2} \hat{z}$ and integrate over the cross section of the dielectric between the conductors to find $P = VI$, the same as with circuit theory.

**Thanks to all............**